Deadbeat control: construction via sets

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Abstract

A geometric generalization of the discrete-time linear deadbeat control problem is studied. The proposed method to generate a deadbeat tracker for a given nonlinear system is constructive and makes use of sets that can be computed iteratively. For demonstration, derivations of deadbeat feedback law and tracker dynamics are provided for various example systems. Based on the method, a simple algorithm that computes the deadbeat gain for a linear system with scalar input is given.

1 Introduction

Deadbeat control (regulation) problem for the discrete-time linear system

$$\hat{x}^+ = A\hat{x} + Bu \tag{1}$$

concerns with finding a (linear) feedback law $u = -K\hat{x}$ such that any solution of the closed-loop system hits the origin in finite time. Thanks to linearity, the same feedback gain K can be used to make the solution $\hat{x}(\cdot)$ of system (1) (exactly) converge to the solution $x(\cdot)$ of the autonomous system $x^+ = Ax$ in finite time. Note that this time $u = K(x - \hat{x})$. The problem being solved in this case is deadbeat tracking. Though deadbeat regulation and deadbeat tracking are equivalent problems for linear systems, the latter subsumes the former when the systems are nonlinear. In this paper we interest ourselves with the nonlinear deadbeat tracking problem. Namely, given two systems, one of them autonomous and the other with a control input, we attempt to find a method to generate a feedback law to couple two systems so that their solutions become equal after some finite time. Our point of departure for generalization however is not (1) but a slight modification of it. Namely,

$$\hat{x}^+ = A(\hat{x} + Bu) \tag{2}$$

which is the form[1] we adopt for generalization. The main reason is that the tools we use in our analysis suggest (2) as a more natural choice.

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[1] Forms (1) and (2) are equivalent from the deadbeat control point of view. See Remark 1.

Deadbeat control theory for linear systems is acknowledged to have been well-established [11, 4]. Different formulations provided different techniques to compute the deadbeat feedback gain [2, 6, 13]. As for discrete-time nonlinear systems, the problem seems to have attracted fewer researchers. Among the cases being studied are bilinear systems [3], polynomial systems [10, 9], and, as a subclass of the latter, Wiener-Hammerstein systems [8].

The toy example that we keep in the back of our mind while we attempt to reach a generalization is the simple case where A is a rotation matrix in $\mathbb{R}^2$

$$A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

with angle of rotation θ different from 0 and π. Letting $B = [1 \quad 0]^T$, the deadbeat tracker turns out to be

$$\hat{x}^+ = A(\hat{x} + B[1 \quad -\cot\theta](x - \hat{x}))$$

Now we state the key observation in this paper: The term in brackets is the intersection of two equivalence classes (sometimes called congruence classes [5]). Namely,

$$\hat{x} + B[1 \quad -\cot\theta](x - \hat{x}) = (\hat{x} + \mathrm{range}(B)) \cap (x + A^{-1}\mathrm{range}(B))$$

as shown in Fig. 1. Based on this observation, the contribution of this paper is

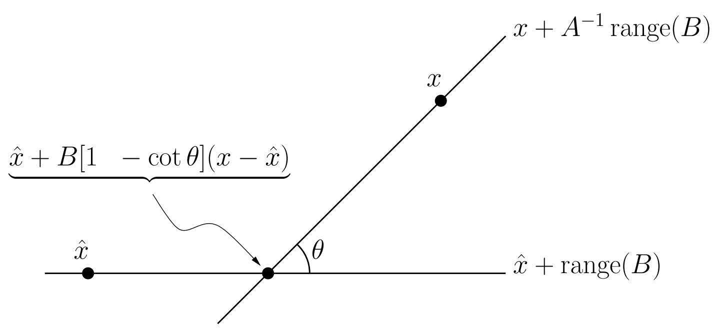

Figure 1: Intersection of two equivalence classes.

intended to be in showing that such equivalence classes can be defined even for nonlinear systems of arbitrary order, which in turn allows one to construct deadbeat feedback laws and hence deadbeat trackers provided that certain conditions (Assumption 1 and Assumption 2) hold. We now note and later demonstrate that when the system is linear those assumptions are minimal for a deadbeat observer to exist. We note that the approach in this paper is the *dual* of the approach adopted in [14].

The remainder of the paper is organized as follows. Next section contains some preliminary material. In Section 3 we give the formal problem definition.

Section 4 is where we describe the sets that we use in construction of the deadbeat tracker. We state and prove the main result in Section 5. We provide examples in Section 6, where we construct deadbeat observers for two different third order systems. In Section 7 we present an algorithm to compute the deadbeat control gain for a linear system with scalar input.

2 Preliminaries

Identity matrix is denoted by I. Null space and range space of a matrix $M \in \mathbb{R}^{m\times n}$ are denoted by $\mathcal{N}(M)$ and $\mathcal{R}(M)$, respectively. Given map $f : \mathcal{X} \to \mathcal{X}$, $f^{-1}(\cdot)$ denotes the *inverse* map in the general sense that for $\mathcal{S} \subset \mathcal{X}$, $f^{-1}(\mathcal{S})$ is the set of all $x \in \mathcal{X}$ satisfying $f(x) \in \mathcal{S}$. That is, we will not need f be bijective when talking about its inverse. Linear map $x \mapsto Ax$ will not be exempt from this notation. Therefore, unless otherwise stated, the reader should not assume that A is a nonsingular matrix when we write A^{-1}. The set of nonnegative integers is denoted by $\mathbb{N}$ and $\mathbb{R}_{>0}$ denotes the set of strictly positive real numbers.

3 Problem definition

Given maps $f : \mathcal{X} \to \mathcal{X}$ and $\mu : \mathcal{X}\times\mathcal{U} \to \mathcal{X}$, consider the following discrete-time system

$$\hat{x}^{+} = f(\mu(\hat{x},\, u)) \tag{3}$$

where $\hat{x} \in \mathcal{X} \subset \mathbb{R}^n$ is the *state* and $u \in \mathcal{U} \subset \mathbb{R}^m$ is the *(control) input*. Map μ is assumed to satisfy

$$x \in \mu(x, \mathcal{U}) \qquad \forall x \in \mathcal{X}\,. \tag{4}$$

Notation $\hat{x}^+$ denotes the state at the next time instant. The goal is to make system (3) (by choosing proper input values) follow the autonomous system $x^+ = f(x)$ in deadbeat fashion. We suppose that we have access to the full state information x of the autonomous system. Then the problem becomes to construct some feedback law $\kappa : \mathcal{X} \times \mathcal{X} \rightrightarrows \mathcal{U}$ such that the states of the below coupled systems

$$\begin{aligned} x^{+} &= f(x) && \text{(5a)}\\ \hat{x}^{+} &\in f(\mu(\hat{x},\, \kappa(\hat{x},\, x))) && \text{(5b)} \end{aligned}$$

converge to each other in finite time. The *solution* of system (5a) at time $k \in \mathbb{N}$, having started at initial condition $x \in \mathcal{X}$, is denoted by $\phi(k,\, x)$. Note that $\phi(0,\, x) = x$ and $\phi(k + 1,\, x) = f(\phi(k,\, x))$ for all x and k. A solution of system (5b) is denoted by $\psi(k,\, \hat{x},\, x)$. For the formal problem description we need the definition below.

Definition 1 *Map $\kappa : \mathcal{X} \times \mathcal{X} \rightrightarrows \mathcal{U}$ is said to be a* deadbeat feedback law *for* system (3) *if there exists $p \geq 1$ such that all solutions of coupled systems* (5) *satisfy*

$$\psi(k, \hat{x}, x) = \phi(k, x)$$

for all $x, \hat{x} \in \mathcal{X}$ and $k \geq p$. System (5b) *then is said to be a* deadbeat tracker*.*

Definition 2 *System* (3) *is said to be* deadbeat controllable *if there exists a deadbeat feedback law for it.*

In this paper we present a procedure to construct a deadbeat tracker from system (3) provided that certain conditions (Assumption 1 and Assumption 2) hold. Our construction will make use of some sets, which we define in the next section. Before moving on into the next section, however, we choose to remind the reader of a standard fact regarding the controllability of linear systems. Then we provide a Lemma 1 as a geometric equivalent of that well-known result. Lemma 1 will find use later when we attempt to interpret and display the generality of the assumptions we will have made.

The following criterion, known as Popov-Belevitch-Hautus (PBH) test, is an elegant tool for checking (deadbeat) controllability.

Proposition 1 (PBH test) *System* (1) *with $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ is deadbeat controllable if and only if*

$$\operatorname{rank}\,[A - \lambda I \quad B] = n \quad \textit{for all} \quad \lambda \neq 0 \tag{6}$$

where λ is a complex scalar.

Remark 1 *From PBH test it readily follows that system* (2) *is deadbeat controllable if and only if system* (1) *is deadbeat controllable. In particular, if $K \in \mathbb{R}^{m\times n}$ is a deadbeat feedback gain for system* (2) *then KA is a deadbeat feedback gain for system* (1)*.*

The below result is a geometric equivalent of PBH test [1, 7].

Lemma 1 *Given $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$, let subspace $\mathcal{S}_{-k}$ of $\mathbb{R}^n$ be defined as $\mathcal{S}_{-k-1} := A^{-1}\mathcal{S}_{-k} + \mathcal{S}_0$ for $k = 0, 1, \ldots$ with $\mathcal{S}_0 := \mathcal{R}(B)$. Then system* (2) *is deadbeat controllable if and only if*

$$\mathcal{S}_{-n} = \mathbb{R}^n\,. \tag{7}$$

Proof. For simplicity we provide the demonstration for the case where each $\mathcal{S}_{-k}$ is a subspace of $\mathbb{C}^n$ (over field $\mathbb{C}$). The case $\mathcal{S}_{-k} \subset \mathbb{R}^n$ is a little longer to prove yet it is true.

We first show (7) $\Longrightarrow$ (6). Suppose (6) fails. That is, there exists a (left) eigenvector $w \in \mathbb{C}^n$ and a nonzero eigenvalue $\lambda \in \mathbb{C}$ such that $w^T A = \lambda w^T$ and $w^T B = 0$. Now suppose for some k we have $w \perp \mathcal{S}_{-k}$. That is, $w^T v = 0$ for

all $v \in \mathcal{S}_{-k}$. We claim that $w \perp A^{-1}\mathcal{S}_{-k}$. Suppose not. Then one can find $v \in A^{-1}\mathcal{S}_{-k}$ such that

$$w^T v \neq 0\,. \tag{8}$$

Also, since $Av \in \mathcal{S}_{-k}$ we can write

$$\begin{aligned} 0 &= w^T A v \\ &= \lambda w^T v \end{aligned}$$

which contradicts (8) since $\lambda \neq 0$. Hence our claim holds. Moreover, since $w^T B = 0$, we can write $w \perp \mathcal{S}_0$. Consequently, $w \perp A^{-1}\mathcal{S}_{-k} + \mathcal{S}_0 = \mathcal{S}_{-k-1}$. We have established therefore

$$w \perp \mathcal{S}_{-k} \implies w \perp \mathcal{S}_{-k-1}\,. \tag{9}$$

Recall that $w \perp \mathcal{S}_0$. That means by (9) that $w \perp \mathcal{S}_{-k}$ for all k. Hence (7) fails.

Now we demonstrate the other direction (6) $\implies$ (7). Note first that for any subspace $\mathcal{S}$ we can write $(A^{-1}\mathcal{S})^\perp = A^T\mathcal{S}^\perp$. Therefore equation $\mathcal{S}_{-k-1} = A^{-1}\mathcal{S}_{-k} + \mathcal{S}_0$ yields

$$\mathcal{S}_{-k-1}^\perp = A^T \mathcal{S}_{-k}^\perp \cap \mathcal{S}_0^\perp\,. \tag{10}$$

Then, since

$$\mathcal{S}_{-k} = \mathcal{R}(B) + A^{-1}\mathcal{R}(B) + A^{-2}\mathcal{R}(B) + \ldots + A^{-k}\mathcal{R}(B)$$

we have $\mathcal{S}_{-k-1} \supset \mathcal{S}_{-k}$. As a result, $\dim \mathcal{S}_{-k-1} \geq \dim \mathcal{S}_{-k}$ for all k. Let us now suppose (7) fails. That means $\dim \mathcal{S}_{-n} \leq n-1$, which implies that there exists $\ell \in \{0, 1, \ldots, n-1\}$ such that $\dim \mathcal{S}_{-\ell-1} = \dim \mathcal{S}_{-\ell} \leq n-1$. Since $\mathcal{S}_{-\ell-1} \supset \mathcal{S}_{-\ell}$, both $\mathcal{S}_{-\ell-1}$ and $\mathcal{S}_{-\ell}$ having the same dimension implies $\mathcal{S}_{-\ell-1} = \mathcal{S}_{-\ell}$. By (10) we can therefore write $\mathcal{S}_{-\ell}^\perp = A^T\mathcal{S}_{-\ell}^\perp \cap \mathcal{S}_0^\perp$, which implies $\mathcal{S}_{-\ell}^\perp \subset A^T\mathcal{S}_{-\ell}^\perp$. Since $\dim \mathcal{S}_{-\ell}^\perp \geq \dim A^T\mathcal{S}_{-\ell}^\perp$ we deduce that $\mathcal{S}_{-\ell}^\perp = A^T\mathcal{S}_{-\ell}^\perp$. Recall that $\dim \mathcal{S}_{-\ell} \leq n-1$. Therefore $\dim \mathcal{S}_{-\ell}^\perp \geq 1$. Then equality $\mathcal{S}_{-\ell}^\perp = A^T\mathcal{S}_{-\ell}^\perp$ implies that there exists an eigenvector $w \in \mathcal{S}_{-\ell}^\perp$ and a nonzero eigenvalue $\lambda \in \mathbb{C}$ such that $w^T A = \lambda w^T$. Note also that $w^T B = 0$ because $\mathcal{S}_{-\ell}^\perp \subset \mathcal{S}_0^\perp = \mathcal{N}(B^T)$. Hence (6) fails. ■

Remark 2 *It is clear from the proof that if* (7) *fails then* $\dim \mathcal{S}_{-k} \leq n-1$ *for all* k*.*

4 Sets

In this section we define certain sets (more formally, *equivalence classes*) associated with system (3). For $x \in \mathcal{X}$ we define

$$[x]_0 := \mu(x, \mathcal{U})\,.$$

Note that for system (2) we have $\mu(x,\,u)=x+Bu$ and $[x]_0=x+\mathcal{R}(B)$. We then let for $k=0,\,1,\,\ldots$

$$[x]_{-k-1}:=\mu([x]^-_{-k},\,\mathcal{U})$$

where

$$[x]^-_{-k}:=f^{-1}([f(x)]_{-k})\,.$$

Remark 3 *Note that $[x]_{-k-1}\supset[x]_{-k}$ and $[x]^-_{-k-1}\supset[x]^-_{-k}$ for all x and k.*

The following two assumptions will be invoked in our main theorem. In hope of making them appear somewhat meaningful and revealing their generality we provide the conditions that they would boil down to for linear systems.

Assumption 1 *There exists $p\geq 1$ such that $[x]_{1-p}=\mathcal{X}$ for all $x\in\mathcal{X}$.*

Assumption 1 is equivalent to deadbeat controllability for linear systems. Below result formalizes this.

Theorem 1 *Linear system* (2) *is deadbeat controllable if and only if Assumption 1 holds.*

Proof. Let $\mathcal{S}_{-k}$ for $k=0,\,1,\,\ldots$ be defined as in Lemma 1. We claim the following.

$$[x]_{-k}=x+\mathcal{S}_{-k}\implies[x]_{-k-1}=x+\mathcal{S}_{-k-1}\,.$$

To see that we write

$$\begin{aligned}
[x]_{-k-1} &= A^{-1}[Ax]_{-k}+\mathcal{S}_0\\
&= A^{-1}(Ax+\mathcal{S}_{-k})+\mathcal{S}_0\\
&= A^{-1}Ax+A^{-1}\mathcal{S}_{-k}+\mathcal{S}_0\\
&= x+\mathcal{N}(A)+A^{-1}\mathcal{S}_{-k}+\mathcal{S}_0\\
&= x+A^{-1}\mathcal{S}_{-k}+\mathcal{S}_0\\
&= x+\mathcal{S}_{-k-1}
\end{aligned}$$

where we used the fact $A^{-1}\mathcal{S}\supset\mathcal{N}(A)$ for any subspace $\mathcal{S}$. Hence our claim holds. Note that $[x]_0=x+\mathcal{S}_0$. Therefore, by induction, $[x]_{-k}=x+\mathcal{S}_{-k}$ for all k.

Now suppose that the system is deadbeat controllable. Then, since $[x]_{-k}=x+\mathcal{S}_{-k}$, we see that Assumption 1 holds with $p=n+1$ thanks to Lemma 1. If however the system is not deadbeat controllable, then by Remark 2 $\dim\mathcal{S}_{-k}\leq n-1$ for all k. Hence Assumption 1 must fail. ∎

Assumption 2 *$\hat{x}\in[x]_0$ implies $[\hat{x}]_0=[x]_0$ for all $x,\,\hat{x}\in\mathcal{X}$.*

Theorem 2 *Assumption 2 comes for free for linear system* (2)*.*

Proof. Evident. ■

Last we let $[x]_1^- := x$ and define map $\pi : \mathcal{X} \times \mathcal{X} \to \{2-p,\, \ldots,\, -1,\, 0,\, 1\}$ as

$$\pi(\hat{x},\, x) := \max\{2-p,\, \ldots,\, -1,\, 0,\, 1\} \quad \text{subject to} \quad [\hat{x}]_0 \cap [x]^-_{\pi(\hat{x},\, x)} \neq \emptyset$$

where p is as in Assumption 1.

5 The result

Below is our main theorem.

Theorem 3 *Suppose Assumptions 1-2 hold. Then system*

$$\hat{x}^+ \in f([\hat{x}]_0 \cap [x]^-_{\pi(\hat{x},\, x)})$$

is a deadbeat tracker.

Proof. We claim the following.

$$\hat{x} \in [x]_{-\ell-1} \implies \hat{x}^+ \in [f(x)]_{-\ell} \tag{11}$$

for all $\ell \in \{0,\, 1,\, \ldots,\, p-2\}$. Let us prove our claim. Note that $\hat{x} \in [x]_{-\ell-1}$ means $\hat{x} \in \mu([x]^-_{-\ell}, \mathcal{U})$, which implies that there exists

$$\eta \in [x]^-_{-\ell} \tag{12}$$

such that $\hat{x} \in \mu(\eta, \mathcal{U}) = [\eta]_0$. By Assumption 2 we have $[\hat{x}]_0 = [\eta]_0$. Then (4) yields

$$\eta \in [\hat{x}]_0\,. \tag{13}$$

From (12) and (13) we have $[\hat{x}]_0 \cap [x]^-_{-\ell} \neq \emptyset$. Therefore $\pi(\hat{x},\, x) \geq -\ell$. Employing Remark 3 we can write

$$\begin{aligned}
\hat{x}^+ &\in f([\hat{x}]_0 \cap [x]^-_{\pi(\hat{x},\, x)}) \\
&\subset f([\hat{x}]_0 \cap [x]^-_{-\ell}) \\
&\subset f([x]^-_{-\ell}) \\
&= f(f^{-1}([f(x)]_{-\ell})) \\
&\subset [f(x)]_{-\ell}\,.
\end{aligned}$$

Hence (11) holds. By Assumption 1 we have $\hat{x} \in [x]_{1-p}$ for all $x,\, \hat{x}$. Therefore (11) and Remark 3 imply the existence of $\ell^* \in \{0,\, 1,\, \ldots,\, p-1\}$ such that

$$\psi(k,\, \hat{x},\, x) \in [\phi(k,\, x)]_0 \tag{14}$$

for all $k \geq \ell^*$. We can write $\phi(k,\, x) \in [\psi(k,\, \hat{x},\, x)]_0$ by (14). Consequently

$$\pi(\psi(k,\, \hat{x},\, x),\, \phi(k,\, x)) = 1$$

for all $k \geq \ell^*$. Then we deduce $\psi(k,\, \hat{x},\, x) = \phi(k,\, x)$ for all $k \geq p$. ■

Corollary 1 *Consider linear system (2) with $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times 1}$. Suppose (A, B) is a controllable[2] pair. Let $\mathcal{S}_{-k}$ for $k = 0, 1, \ldots$ be defined as in Lemma 1. Then system*

$$\hat{x}^+ = A((\hat{x} + \mathcal{S}_0) \cap (x + A^{-1}\mathcal{S}_{2-n}))$$

is a deadbeat tracker.

6 Examples

Here, for two third order nonlinear systems, we construct deadbeat trackers. In the first example we study a simple homogeneous system and show that the construction yields a homogeneous feedback law. Hence our method may be thought to be somewhat *natural* in the vague sense that the tracker it generates inherits certain intrinsic properties of the system. In the second example we aim to provide a demonstration on tracker construction for a system that resides in a state space different than $\mathbb{R}^n$.

6.1 Homogeneous system

Consider system (3) with

$$f(x) := \begin{bmatrix} -x_2 \\ x_1 + x_3^{1/3} \\ x_2^3 + x_3 \end{bmatrix} \quad \text{and} \quad \mu(x,\, u) := \begin{bmatrix} x_1 \\ x_2 \\ x_3 + u^3 \end{bmatrix}$$

where $x = [x_1\ x_2\ x_3]^T$. Let $\mathcal{X} = \mathbb{R}^3$ and $\mathcal{U} = \mathbb{R}$. If we let dilation Δ_λ be

$$\Delta_\lambda := \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda^3 \end{bmatrix}$$

with $\lambda \in \mathbb{R}$, then we realize that

$$f(\Delta_\lambda x) = \Delta_\lambda f(x) \quad \text{and} \quad \mu(\Delta_\lambda x,\, \lambda u) = \Delta_\lambda \mu(x,\, u)\,.$$

That is, the system is homogeneous [12] with respect to dilation Δ. Before describing the relevant sets $[x]_{-k}$ and $[x]^-_{-k}$ we want to mention that f is bijective and its inverse is

$$f^{-1}(x) = \begin{bmatrix} x_2 - (x_1^3 + x_3)^{1/3} \\ -x_1 \\ x_1^3 + x_3 \end{bmatrix} \tag{15}$$

Now we are ready to construct our sets. By definition $[x]_0 = \mu(x, \mathcal{U})$. Therefore we can write

$$[x]_0 = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ \alpha^3 \end{bmatrix} : \alpha \in \mathbb{R} \right\}$$

[2] That is, rank $[B\ AB\ \ldots\ A^{n-1}B] = n$.

By (15) we can then proceed as

$$\begin{aligned}
[x]_0^- &= f^{-1}([f(x)]_0) \\
&= f^{-1}\left(\left\{\begin{bmatrix} -x_2 \\ x_1 + x_3^{1/3} \\ \alpha^3 \end{bmatrix} : \alpha \in \mathbb{R}\right\}\right) \\
&= \left\{ f^{-1}\left(\begin{bmatrix} -x_2 \\ x_1 + x_3^{1/3} \\ \alpha^3 \end{bmatrix}\right) : \alpha \in \mathbb{R}\right\} \\
&= \left\{\begin{bmatrix} x_1 + x_3^{1/3} + (x_2^3 - \alpha^3)^{1/3} \\ x_2 \\ -x_2^3 + \alpha^3 \end{bmatrix} : \alpha \in \mathbb{R}\right\}
\end{aligned}$$

Since $[x]_{-1} = \mu([x]_0^-, \mathcal{U})$ we can write

$$\begin{aligned}
[x]_{-1} &= \left\{\begin{bmatrix} x_1 + x_3^{1/3} + (x_2^3 - \alpha^3)^{1/3} \\ x_2 \\ \beta^3 \end{bmatrix} : \alpha,\, \beta \in \mathbb{R}\right\} \\
&= \left\{\begin{bmatrix} \alpha \\ x_2 \\ \beta^3 \end{bmatrix} : \alpha,\, \beta \in \mathbb{R}\right\}
\end{aligned}$$

We can now construct $[x]_{-1}^-$ as

$$\begin{aligned}
[x]_{-1}^- &= f^{-1}([f(x)]_{-1}) \\
&= \left\{ f^{-1}\left(\begin{bmatrix} \alpha \\ x_1 + x_3^{1/3} \\ \beta^3 \end{bmatrix}\right) : \alpha,\, \beta \in \mathbb{R}\right\} \\
&= \left\{\begin{bmatrix} x_1 + x_3^{1/3} - (\alpha^3 + \beta^3)^{1/3} \\ -\alpha \\ \alpha^3 + \beta^3 \end{bmatrix} : \alpha,\, \beta \in \mathbb{R}\right\} \\
&= \left\{\begin{bmatrix} x_1 + x_3^{1/3} - \beta \\ \alpha \\ \beta^3 \end{bmatrix} : \alpha,\, \beta \in \mathbb{R}\right\}
\end{aligned}$$

Now it is easy to see that $\mu([x]_{-1}^-, \mathbb{R}) = \mathbb{R}^3$. Therefore $[x]_{-2} = \mu([x]_{-1}^-, \mathcal{U}) = \mathcal{X}$ and Assumption 1 is satisfied with $p = 3$. Observe also that the following intersection

$$\begin{aligned}
[\hat{x}]_0 \cap [x]_{-1}^- &= \left\{\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \gamma^3 \end{bmatrix} : \gamma \in \mathbb{R}\right\} \cap \left\{\begin{bmatrix} x_1 + x_3^{1/3} - \beta \\ \alpha \\ \beta^3 \end{bmatrix} : \alpha,\, \beta \in \mathbb{R}\right\} \\
&= \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ (x_1 - \hat{x}_1 + x_3^{1/3})^3 \end{bmatrix}
\end{aligned}$$

is singleton, which means that $[\hat{x}]_0 \cap [x]^-_{\pi(\hat{x},\, x)} = [\hat{x}]_0 \cap [x]^-_{-1}$ for all $\hat{x}$ and x. The dynamics of the deadbeat observer then read

$$\begin{aligned} \hat{x}^+ &= f([\hat{x}]_0 \cap [x]^-_{-1}) \\ &= \begin{bmatrix} -\hat{x}_2 \\ x_1 + x_3^{1/3} \\ \hat{x}_2^3 + (x_1 - \hat{x}_1 + x_3^{1/3})^3 \end{bmatrix} \end{aligned}$$

In particular, solving for u in $\mu(\hat{x},\, u) = [\hat{x}]_0 \cap [x]^-_{-1}$ readily yields $u = ((x_1 - \hat{x}_1 + x_3^{1/3})^3 - \hat{x}_3)^{1/3}$. Therefore

$$\kappa(\hat{x},\, x) := ((x_1 - \hat{x}_1 + x_3^{1/3})^3 - \hat{x}_3)^{1/3}$$

is a deadbeat feedback law. Notice that

$$\kappa(\Delta_\lambda \hat{x},\, \Delta_\lambda x) = \lambda \kappa(\hat{x},\, x)\,.$$

That is, the feedback law and hence the tracker are homogeneous with respect to dilation Δ.

6.2 Positive system

Consider system (3) with

$$f(x) := \begin{bmatrix} x_1 x_2 x_3 \\ x_3/x_1 \\ \sqrt{x_1 x_2} \end{bmatrix} \quad \text{and} \quad \mu(x,\, u) := \begin{bmatrix} x_1/u \\ x_2 u^2 \\ x_3/u \end{bmatrix}$$

Let $\mathcal{X} = \mathbb{R}^3_{>0}$ and $\mathcal{U} = \mathbb{R}_{>0}$. We construct the relevant sets $[x]_{-k}$ and $[x]^-_{-k}$ as follows. We begin with $[x]_0$.

$$[x]_0 = \left\{ \begin{bmatrix} x_1/\alpha \\ x_2 \alpha^2 \\ x_3/\alpha \end{bmatrix} : \alpha > 0 \right\}$$

Note that

$$f^{-1}(x) = \begin{bmatrix} x_1/(x_2 x_3^2) \\ x_2 x_3^4/x_1 \\ x_1/x_3^2 \end{bmatrix}$$

Therefore

$$
\begin{aligned}
[x]_0^- &= f^{-1}([f(x)]_0) \\
&= f^{-1}\left(\left\{\begin{bmatrix} x_1x_2x_3/\alpha \\ x_3\alpha^2/x_1 \\ \sqrt{x_1x_2}/\alpha \end{bmatrix} : \alpha > 0\right\}\right) \\
&= \left\{f^{-1}\left(\begin{bmatrix} x_1x_2x_3/\alpha \\ x_3\alpha^2/x_1 \\ \sqrt{x_1x_2}/\alpha \end{bmatrix}\right) : \alpha > 0\right\} \\
&= \left\{\begin{bmatrix} x_1/\alpha \\ x_2/\alpha \\ x_3\alpha \end{bmatrix} : \alpha > 0\right\}
\end{aligned}
$$

Since $[x]_{-1} = \mu([x]_0^-, \mathcal{U})$ we can write

$$
[x]_{-1} = \left\{\begin{bmatrix} x_1/(\alpha\beta) \\ x_2\beta^2/\alpha \\ x_3\alpha/\beta \end{bmatrix} : \alpha,\, \beta > 0\right\}
$$

We can now construct $[x]_{-1}^-$ as

$$
\begin{aligned}
[x]_{-1}^- &= f^{-1}([f(x)]_{-1}) \\
&= \left\{f^{-1}\left(\begin{bmatrix} x_1x_2x_3/(\alpha\beta) \\ x_3\beta^2/(x_1\alpha) \\ \alpha\sqrt{x_1x_2}/\beta \end{bmatrix}\right) : \alpha,\, \beta > 0\right\} \\
&= \left\{\begin{bmatrix} x_1/(\alpha^2\beta) \\ x_2\alpha^4/\beta \\ x_3\beta/\alpha^3 \end{bmatrix} : \alpha,\, \beta > 0\right\}
\end{aligned}
$$

Now, it can be shown that $\mu([x]_{-1}^-, \mathbb{R}_{>0}) = \mathbb{R}^3_{>0}$. Therefore $[x]_{-2} = \mu([x]_{-1}^-, \mathcal{U}) = \mathcal{X}$ and Assumption 1 is satisfied with $p = 3$. Observe also that the following intersection

$$
\begin{aligned}
[\hat{x}]_0 \cap [x]_{-1}^- &= \left\{\begin{bmatrix} \hat{x}_1/\gamma \\ \hat{x}_2\gamma^2 \\ \hat{x}_3/\gamma \end{bmatrix} : \gamma > 0\right\} \cap \left\{\begin{bmatrix} x_1/(\alpha^2\beta) \\ x_2\alpha^4/\beta \\ x_3\beta/\alpha^3 \end{bmatrix} : \alpha,\, \beta > 0\right\} \\
&= \begin{bmatrix} \hat{x}_1^{4/3}\hat{x}_2^{5/3}\hat{x}_3^2/(x_1^{1/3}x_2^{5/3}x_3^2) \\ x_1^{2/3}x_2^{10/3}x_3^4/(\hat{x}_1^{2/3}\hat{x}_2^{7/3}\hat{x}_3^4) \\ \hat{x}_1^{1/3}\hat{x}_2^{5/3}\hat{x}_3^3/(x_1^{1/3}x_2^{5/3}x_3^2) \end{bmatrix}
\end{aligned}
$$

is singleton, which means that $[\hat{x}]_0 \cap [x]_{\pi(\hat{x},\, x)}^- = [\hat{x}]_0 \cap [x]_{-1}^-$ for all $\hat{x}$ and x. The dynamics of the deadbeat observer then read

$$
\begin{aligned}
\hat{x}^+ &= f([\hat{x}]_0 \cap [x]_{-1}^-) \\
&= \begin{bmatrix} \hat{x}_1\hat{x}_2\hat{x}_3 \\ \hat{x}_3/\hat{x}_1 \\ \hat{x}_1^{1/3}x_1^{1/6}x_2^{5/6}x_3/(\hat{x}_2^{1/3}\hat{x}_3) \end{bmatrix}
\end{aligned}
$$

In particular, $u = x_1^{1/3} x_2^{5/3} x_3^2 / (\hat{x}_1^{1/3} \hat{x}_2^{5/3} \hat{x}_3^2)$ solves $\mu(\hat{x}, u) = [\hat{x}]_0 \cap [x]_{-1}^-$. Therefore

$$\kappa(\hat{x}, x) := x_1^{1/3} x_2^{5/3} x_3^2 / (\hat{x}_1^{1/3} \hat{x}_2^{5/3} \hat{x}_3^2)$$

is a deadbeat feedback law.

7 An algorithm for deadbeat gain

In this section we provide an algorithm to compute the deadbeat feedback gain for a linear system (1) with scalar input. (The algorithm follows from Corollary 1.) Namely, given a controllable pair (A, B) with $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times 1}$, we provide a procedure to compute the gain $K \in \mathbb{R}^{1\times n}$ that renders matrix $A - BK$ nilpotent. Below we let $\mathrm{null}(\cdot)$ be some function such that, given matrix $M \in \mathbb{R}^{m\times n}$ whose dimension of null space is k, $\mathrm{null}(M)$ is some $n \times k$ matrix whose columns span the null space of M.

Algorithm 1 *Given $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times 1}$, the following algorithm generates deadbeat gain $K \in \mathbb{R}^{1\times n}$.*

$$\begin{aligned}
&X = B\\
&\textbf{for}\quad i = 1 : n-2\\
&\qquad X = [A^{-1}X \quad B]\\
&\textbf{end}\\
&K_2 = \frac{\mathrm{null}((A^{-1}X)^T)^T}{\mathrm{null}((A^{-1}X)^T)^T B}\\
&K = K_2 A
\end{aligned}$$

Remark 4 *Matrix K_2 appearing in Algorithm 1 is the deadbeat gain for system* (2)*. That is, matrix $A(I - BK_2)$ is nilpotent.*

Recall that for any subspace $\mathcal{S}$ we can write $(A^{-1}\mathcal{S})^\perp = A^T \mathcal{S}^\perp$. Therefore for the case when A is not invertible or when matrix inversion is costly one can use the below *dual* algorithm.

Algorithm 2 *Given $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times 1}$, the following algorithm generates deadbeat gain $K \in \mathbb{R}^{1\times n}$.*

$$\begin{aligned}
&X_{\mathrm{perp}} = \mathrm{null}(B^T)\\
&\textbf{for}\quad i = 1 : n-2\\
&\qquad X_{\mathrm{perp}} = \mathrm{null}([\mathrm{null}((A^T X_{\mathrm{perp}})^T) \quad B]^T)\\
&\textbf{end}\\
&K_2 = \frac{(A^T X_{\mathrm{perp}})^T}{(A^T X_{\mathrm{perp}})^T B}\\
&K = K_2 A
\end{aligned}$$

For the interested reader we below give MATLAB codes. Algorithm 1 can be realized through the following lines.

```
X = B;
for i = 1:n-2
    X = [A^(-1)*X B];
end
Ktwo = null((A^(-1)*X)')'/(null((A^(-1)*X)')'*B);
K = Ktwo*A;
```

Likewise, Algorithm 2 can be coded as follows.

```
Xperp = null(B');
for i = 1:n-2
    Xperp = null([null((A'*Xperp)') B]');
end
Ktwo = (A'*Xperp)'/((A'*Xperp)'*B);
K = Ktwo*A;
```

8 Conclusion

For nonlinear systems a method to construct a deadbeat tracker is proposed. The resultant tracker can be considered as a generalization of the linear deadbeat tracker. The construction makes use of sets that are generated iteratively. Through such iterations, deadbeat feedback laws are derived for two academic examples. Also, for computing the deadbeat gain for a linear system with scalar input, an algorithm and its dual are given.

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